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INTEGRABLE MODULES OVER $\widehat{\mathfrak{gl}}_m$ AND THE DOUBLE AFFINE HECKE ALGEBRA

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Introduction

Motivated by conformal field theory on the Riemann sphere, we introduce a certain space of coinvariants obtained from tensor product of representations of the affine Lie algebra $\widehat{\mathfrak{gl}}_m$.

In [AST], an action of the degenerate affine Hecke algebra H_κ is defined on this space through the Knizhnik-Zamolodchikov connection. This construction gives a functor from the category of highest (or lowest) weight modules over $\widehat{\mathfrak{gl}}_m$ to the category of H_κ -modules.

We will see that the integrable $\widehat{\mathfrak{gl}}_m$ -modules correspond by this functor to irreducible H_κ -modules whose structure is described combinatorially. We also focus on the symmetric part of these irreducible H_κ -modules; i.e., the subspace consisting of those elements which are invariant with respect to the action of the Weyl group. We present a spectral decomposition of the symmetric part, and a character formula, which is described by level restricted analogue of the Kostka polynomial.

1. AFFINE LIE ALGEBRA

Throughout this note, we use the notation $[i, j] = \{i, i+1, \dots, j\}$ for $i, j \in \mathbb{Z}$.

Let $m \in \mathbb{Z}_{\geq 2}$. Let \mathfrak{g} denote the Lie algebra \mathfrak{gl}_m consisting of all $n \times n$ -matrices over \mathbb{C} . Let $\mathfrak{g}[t, t^{-1}]$ denote the Lie algebra consisting of all $n \times n$ -matrices over $\mathbb{C}[t, t^{-1}]$. Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c_{\mathfrak{g}}$ be the affine Lie algebra with the commutation relation

$$[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j} + \text{trace}(ab) i \delta_{i+j, 0} c_{\mathfrak{g}}$$

for $a, b \in \mathfrak{g}$, $i, j \in \mathbb{Z}$.

Let \mathfrak{h} denote the Cartan subalgebra of \mathfrak{g} consisting of all diagonal matrices, and let \mathfrak{h}^* denote its dual space. A Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ is given by $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_{\mathfrak{g}}$. Its dual space is denoted by $\widehat{\mathfrak{h}}^*$. We regard $\widehat{\mathfrak{h}}^*$ as a subspace of $\widehat{\mathfrak{h}}^*$ through the identification $\widehat{\mathfrak{h}}^* \cong \mathfrak{h}^* \oplus \mathbb{C}c_{\mathfrak{g}}^*$.

Fix $\ell \in \mathbb{C}$. For $\lambda \in \mathfrak{h}^*$, $\widehat{M}_\ell(\lambda)$ denote the highest weight Verma module of highest weight $\lambda + \ell c_{\mathfrak{g}}^* \in \widehat{\mathfrak{h}}^*$, and let $\widehat{M}_\ell^\dagger(\lambda)$ denote the lowest weight Verma module of lowest weight $-\lambda - \ell c_{\mathfrak{g}}^* \in \widehat{\mathfrak{h}}^*$. Their irreducible quotients are denoted by $\widehat{L}_\ell(\lambda)$ and $\widehat{L}_\ell^\dagger(\lambda)$ respectively.

A $\widehat{\mathfrak{g}}$ -module M is said to be of level ℓ if c acts as a scalar ℓ . For example, $\widehat{M}_\ell(\lambda)$ and $\widehat{L}_\ell(\lambda)$ are of level ℓ , and $\widehat{M}_\ell^\dagger(\lambda)$ and $\widehat{L}_\ell^\dagger(\lambda)$ are of level $-\ell$.

We identify \mathfrak{h} with \mathbb{C}^m , and introduce its subspaces $X_m = \mathbb{Z}^m$ and

$$X_m^+ = \{(\lambda_1, \dots, \lambda_m) \in X_m \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m\},$$

$$X_m^+(\ell) = \{(\lambda_1, \dots, \lambda_m) \in X_m^+ \mid \lambda_1 - \lambda_m \leq \ell\}.$$

Note that $\widehat{L}_\ell(\lambda)$ and $\widehat{L}_\ell^\dagger(\lambda)$ are integrable for $\lambda \in X_m^+(\ell)$, and that $X_m^+(\ell)$ is empty unless $\ell \in \mathbb{Z}_{\geq 0}$.

Let $E = \mathbb{C}^m$ denote the vector representation of \mathfrak{g} . Put $E[z, z^{-1}] = E \otimes \mathbb{C}[z, z^{-1}]$, which we regard as a $\mathfrak{g}[t, t^{-1}]$ -module through the correspondence $a \otimes t^k \mapsto a \otimes z^k$.

2. THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $n \in \mathbb{Z}_{\geq 2}$. Let V denote the n -dimensional vector space over \mathbb{C} with the basis $\{y_i\}_{i \in [1, n]}$: $V = \bigoplus_{i \in [1, n]} \mathbb{C}y_i$. Introduce the non-degenerate symmetric bilinear form (\mid) on V by $(y_i \mid y_j) = \delta_{ij}$. Let $V^* = \bigoplus_{i=1}^n \mathbb{C}x_i$ be the dual space of V , where x_i is the dual vector of y_i . The natural pairing is denoted by $\langle \mid \rangle : V^* \times V \rightarrow \mathbb{C}$.

Put $\alpha_{ij} = x_i - x_j$, $\alpha_{ij}^\vee = y_i - y_j$ and $\alpha_i = \alpha_{ii+1}$, $\alpha_i^\vee = \alpha_{ii+1}$. Then $R = \{\alpha_{ij} \mid i, j \in [1, n], i \neq j\}$ and $R^+ = \{\alpha_{ij} \in R \mid i < j\}$ give a set of roots and a set of positive roots of type A_{n-1} respectively.

Let W denote the Weyl group associated with the root system R , which is isomorphic to the symmetric group \mathfrak{S}_n of degree n . Denote by s_α the reflection in W corresponding to $\alpha \in R$. We write $s_i = s_{\alpha_i}$ and $s_{ij} = s_{\alpha_{ij}}$.

Put $P = \bigoplus_{i \in [1, n]} \mathbb{Z}x_i$, which is preserved by W . Define the *extended affine Weyl group* \widehat{W} as the semidirectproduct $P \rtimes W$ with the relation $w\tau_\eta w^{-1} = \tau_{w(\eta)}$, where τ_η denotes the element of \widehat{W} corresponding to $\eta \in P$.

Let $S(V)$ denote the symmetric algebra of V , which can be identified with the polynomial ring $\mathbb{C}[\underline{y}] = \mathbb{C}[y_1, \dots, y_n]$.

Fix $\kappa \in \mathbb{C}$. The *degenerate double affine Hecke algebra* (degenerate DAHA) H_κ of GL_n is an associative \mathbb{C} -algebra generated by the algebra $\mathbb{C}P$, $\mathbb{C}W$ and $S(V)$, and subjects to the following defining relations

([C1]):

$$s_i h = s_i(h) s_i - \langle \alpha_i | h \rangle \quad (i \in [1, n], h \in V),$$

$$s_i e^\eta s_i = e^{s_i(\eta)} \quad (i \in [1, n], \eta \in P),$$

$$[h, e^\eta] = \kappa \langle \eta | h \rangle e^\eta + \sum_{\alpha \in R^+} \langle \alpha | h \rangle \frac{(e^\eta - e^{s_\alpha(\eta)})}{1 - e^{-\alpha}} s_\alpha \quad (h \in V, \eta \in P),$$

where e^η denote the element of $\mathbb{C}P$ corresponding to $\eta \in P$.

It is known that $H_\kappa \cong \mathbb{C}P \otimes \mathbb{C}W \otimes S(V)$ as a vector space. The subalgebra $H^{\text{aff}} = \mathbb{C}W \cdot S(V)$ is called the degenerate affine Hecke algebra. Note that the subalgebra $\mathbb{C}P \cdot \mathbb{C}W$ is isomorphic to $\widehat{\mathbb{C}W}$.

3. INDUCED REPRESENTATIONS OF H_κ

For $\lambda \in X_m = \mathbb{Z}^m$ we write $\lambda \models n$ when $\sum_{i \in [1, m]} \lambda_i = n$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [1, m]$. Let $\lambda, \mu \in X_m$ such that $\lambda - \mu \models n$. Introduce the subalgebra $H_\lambda = \mathbb{C}W_{\lambda-\mu} \cdot S(V)$ of H_κ , where $W_{\lambda-\mu}$ denote the parabolic subgroup $\mathfrak{S}_{\lambda_1-\mu_1} \times \cdots \times \mathfrak{S}_{\lambda_m-\mu_m}$ of W .

Let $\mathbb{C}1_{\lambda, \mu}$ denote the one dimensional representation of $H_{\lambda-\mu}$ such that

$$\begin{aligned} w 1_{\lambda, \mu} &= 1_{\lambda, \mu} \quad (w \in W_{\lambda-\mu}), \\ y_i 1_{\lambda, \mu} &= \langle \zeta_{\lambda, \mu} | y_i \rangle 1_{\lambda, \mu} \quad (i \in [1, n]), \end{aligned}$$

where $\zeta_{\lambda, \mu}$ denote the element of V^* given by

$$(3.1) \quad \langle \zeta_{\lambda, \mu} | y_i \rangle = \mu_j + i - m_j - j - 1 \quad \text{for } i \in [m_j + 1, m_{j+1}],$$

with $m_0 = 0$ and $m_j = \sum_{k \in [1, j]} (\lambda_k - \mu_k)$ ($j \in [1, m]$). Define an H_κ -module by $\mathcal{M}_\kappa(\lambda, \mu) = H_\kappa \otimes_{H_{\lambda-\mu}} \mathbb{C}1_{\lambda, \mu}$. Obviously we have

$$\mathcal{M}_\kappa(\lambda, \mu) \cong \widehat{\mathbb{C}W} / W_{\lambda-\mu} \cong \mathbb{C}P \otimes \mathbb{C}W / W_{\lambda-\mu}$$

as an \widehat{W} -module.

In the rest, we often identify the group ring $\mathbb{C}P$ with the Laurent polynomial ring $\mathbb{C}[\underline{z}^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ via the correspondence $e^{x_i} \mapsto z_i$.

Example 3.1. Let $m = 1$ and let $\lambda = (n)$ and $\mu = (0)$. Then $\mathcal{M}_\kappa(\lambda, \mu) \cong \mathbb{C}P = \mathbb{C}[\underline{z}^{\pm 1}]$, which is called the (Laurent) polynomial representation. On the representation $\mathbb{C}P$, the element y_i ($i \in [1, n]$) acts as the *Cherednik-Dunkl* operator

$$(3.2) \quad T_i = \kappa z_i \frac{\partial}{\partial z_i} + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) + i - 1.$$

The simultaneous eigenvectors of T_1, \dots, T_n are called the nonsymmetric Jack polynomials.

4. THE SPACE OF COINVARIANTS AND THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

Let $\ell \in \mathbb{C}$. Let M be a highest weight module of level ℓ and let N be a lowest weight module of level $-\ell$. We set

$$\begin{aligned}\tilde{\mathcal{C}}(M, N) &= M \otimes E[z_1, z_1^{-1}] \otimes \cdots \otimes E[z_n, z_n^{-1}] \otimes N, \\ \mathcal{C}(M, N) &= \tilde{\mathcal{C}}(M, N) / \mathfrak{g}[t, t^{-1}] \tilde{\mathcal{C}}(M, N).\end{aligned}$$

Let $\sigma_{ij} \in \text{End}_{\mathbb{C}}[\mathbb{C}[z^{\pm 1}]]$ denote the permutation of z_i and z_j . Let $\Omega_{ij} \in \text{End}_{\mathbb{C}}(E^{\otimes n})$ denote the permutation of i -th and j -th component of the tensor product. Note that $\tilde{\mathcal{C}}(M, N) \cong M \otimes E^{\otimes n} \otimes \mathbb{C}[z^{\pm 1}] \otimes N$, through which we regard σ_{ij} and Ω_{ij} as elements in $\text{End}_{\mathbb{C}}(\tilde{\mathcal{C}}(M, N))$.

For $i \in [0, n+1]$, define $\theta_i : \hat{\mathfrak{g}} \rightarrow U(\hat{\mathfrak{g}})^{\otimes n+2}$ by $\theta_i(u) = 1^{\otimes i} \otimes u \otimes 1^{\otimes n-i+1}$. For $i, j \in [0, n+1]$ with $i < j$, define $\theta_{ij} : \hat{\mathfrak{g}}^{\otimes 2} \rightarrow U(\hat{\mathfrak{g}})^{\otimes n+2}$ by $\theta_{ij}(u \otimes v) = 1^{\otimes i} \otimes u \otimes 1^{\otimes j-i-1} \otimes v \otimes 1^{\otimes n-j+1}$.

Let e_{ab} denote the matrix unit of \mathfrak{g} with only non-zero entries 1 at the (a, b) -th component. Put $r = \frac{1}{2} \sum_{a \in [1, m]} e_{aa} \otimes e_{aa} + \sum_{1 \leq a < b \leq m} e_{ab} \otimes e_{ba}$ and put $r_{ij} = \theta_{ij}(r)$.

For $i \in [1, n]$, put

$$(4.1) \quad \hat{r}_{0i} = r_{0i} + \sum_{k \in \mathbb{Z}_{\geq 1}} \sum_{a, b \in [1, m], a \neq b} \theta_{0i}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})),$$

$$(4.2) \quad \hat{r}_{in+1} = r_{in+1} + \sum_{k \in \mathbb{Z}_{\geq 1}} \sum_{a, b \in [1, m], a \neq b} \theta_{in+1}((e_{ab} \otimes t^k) \otimes (e_{ba} \otimes t^{-k})),$$

which are elements of some completion of $U(\mathfrak{g}[t, t^{-1}])^{\otimes n+2}$ and define well-defined operators on $\tilde{\mathcal{C}}(M, N)$.

Define the linear operators on $\tilde{\mathcal{C}}(M, N)$ by

$$\begin{aligned}D_i &= \kappa z_i \frac{\partial}{\partial z_i} + \hat{r}_{0i} - \hat{r}_{in+1} + \sum_{1 \leq j < i} r_{ij} - \sum_{i < j \leq n} r_{ji} + \theta_i(\rho^{\vee}) \\ &\quad + \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - \sigma_{ij}) \Omega_{ij} + i - 1,\end{aligned}$$

where $\rho^{\vee} = \sum_{k \in [1, m]} \frac{1}{2} (n - 2k + 1) e_{aa} \in \mathfrak{h}$.

Theorem 4.1 (Theorem 4.2.2 in [AST]). *Let M be a highest weight module of $\hat{\mathfrak{g}}$ of level $\kappa - m$ and let N be a lowest weight module of level $-\kappa + m$.*

(i) There exists a unique algebra homomorphism $\varpi : H_\kappa^{\text{rat}} \rightarrow \text{End}_{\mathbb{C}}(\tilde{\mathcal{C}}(M, N))$ such that

$$(4.3) \quad \varpi(s_i) = \Omega_{ii+1} \sigma_{ii+1} \quad (i \in [1, n-1]),$$

$$(4.4) \quad \varpi(e^{x_i}) = z_i \quad (i \in [1, n]),$$

$$(4.5) \quad \varpi(y_i) = D_i \quad (i \in [1, n]).$$

(ii) The H_κ -action on $\mathcal{C}(M, N)$ above preserves the subspace $\mathfrak{g}[t, t^{-1}]\tilde{\mathcal{C}}(M, N)$:

$$\varpi(a)\mathfrak{g}[t, t^{-1}]\tilde{\mathcal{C}}(M, N) \subseteq \mathfrak{g}[t, t^{-1}]\tilde{\mathcal{C}}(M, N)$$

for all $a \in H_\kappa$. Therefore, ϖ induces an H_κ -module structure on $\mathcal{C}(M, N)$.

5. IMAGES OF THE FUNCTOR

The following statement has been shown in [AST].

Proposition 5.1 (Proposition 5.3.1 in [AST]). *Let $\kappa \in \mathbb{C}$ and put $\ell = \kappa - m$.*

(i) *Let $\lambda, \mu \in X_m^+$. Then*

$$\mathcal{C}(\widehat{M}_\ell(\mu), \widehat{M}_\ell^\dagger(\lambda)) \cong \begin{cases} \mathcal{M}_\kappa(\lambda, \mu) & \text{if } \lambda - \mu \models n, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Let $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then*

$$\mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda)) \cong \mathcal{C}(\widehat{M}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda)) \cong \mathcal{C}(\widehat{L}_\ell(\mu), \widehat{M}_\ell^\dagger(\lambda)).$$

For each $\lambda \in X_m$, we have the additive functor $F_\lambda(-) = \mathcal{C}(-, \widehat{M}_\ell^\dagger(\lambda))$ from the category of highest weight modules over $\widehat{\mathfrak{g}}$ to the category of H_κ -modules. It is right exact and sends the Verma module $\widehat{M}_\ell(\mu)$ to the induced module $\mathcal{M}_\kappa(\lambda, \mu)$ by Proposition 5.1. In the sequel, we will determine the image $F_\lambda(\widehat{L}_\ell(\mu))$ of the irreducible module $\widehat{L}_\ell(\mu)$ in the case where $\lambda, \mu \in X_m^+(\ell)$. Note that $F_\lambda(\widehat{L}_\ell(\mu)) \cong \mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda))$, and note also that it is a quotient of $F_\lambda(\mathcal{M}_\kappa(\lambda, \mu))$.

Let $\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then it is known that the H_κ -module $\mathcal{M}_\kappa(\lambda, \mu)$ has a unique simple quotient ([AST, S1]), which we will denote by $\mathcal{L}_\kappa(\lambda, \mu)$.

The irreducible modules $\mathcal{L}_\kappa(\lambda, \mu)$ for $\lambda, \mu \in X_m^+(\ell)$ are investigated in [SV], and in particular their structure is described combinatorially using tableaux on periodic skew diagrams. We give a short review of the theory of periodic tableaux and the tableaux representations of H_κ in Appendix. By means of this combinatorial description, we can estimate the kernel of the projection $\mathcal{M}_\kappa(\lambda, \mu) \rightarrow \mathcal{L}_\kappa(\lambda, \mu)$. By comparing it with the kernel of $\mathcal{M}_\kappa(\lambda, \mu) \rightarrow F_\lambda(\widehat{L}_\ell(\mu))$, we have

Theorem 5.2. *Let $\kappa \in \mathbb{Z}_{\geq 1}$ and put $\ell = \kappa - m$. Let $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then the H_κ -module $\mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda))$ is irreducible:*

$$\mathcal{C}(\widehat{L}_\ell(\mu), \widehat{L}_\ell^\dagger(\lambda)) \cong \mathcal{L}_\kappa(\lambda, \mu),$$

and moreover it is semisimple over $S(V)$. (See Theorem A.3 for the combinatorial description of the weight decomposition).

The classification of the irreducible H_κ -modules which are semisimple over $S(V)$ is given in [C2, SV], from which (or from Theorem A.4) we have

Corollary 5.3. *Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let L be an irreducible H_κ -module which is finitely generated and admits a weight decomposition of the form $L = \bigoplus_{\zeta \in P} L_\zeta$, where $L_\zeta = \{v \in L \mid yv = \langle \zeta \mid y \rangle \forall y \in V\}$. Then there exists $m \in [1, n]$ and $\lambda, \mu \in X_m^+(\kappa - m)$ such that $L \cong \mathcal{C}(\widehat{L}_{\kappa-m}(\mu), \widehat{L}_{\kappa-m}^\dagger(\lambda))$.*

6. LOCALIZATION AND CONFORMAL COINVARIANTS

We will see the relation between our space $\mathcal{C}(M, N)$ of coinvariants and the space of conformal coinvariants in Wess-Zumino-Witten model [TK, TUY].

Observe that the group ring $\mathbb{C}P$ can be seen as the coordinate ring $A = \mathbb{C}[\mathcal{T}]$ of the affine variety $\mathcal{T} = (\mathbb{C} \setminus \{0\})^n$. Put $\mathcal{T}_\circ = \mathcal{T} \setminus \Delta$, where $\Delta = \bigcup_{i < j} \{(\xi_1, \dots, \xi_n) \in \mathcal{T} \mid \xi_i/\xi_j = 1\}$, and put $A_\circ = \mathbb{C}[\mathcal{T}_\circ]$. Namely, A_\circ is the localization of A at Δ ; $A_\circ = \mathbb{C} \left[z_1^{\pm 1}, \dots, z_n^{\pm 1}, \frac{1}{1 - z_i/z_j} \ (i < j) \right]$. Let $\mathcal{D}(\mathcal{T}_\circ)$ denote the ring of algebraic differential operators on \mathcal{T}_\circ . Then the Cherednik-Dunkl operators in Example 3.1 T_1, \dots, T_n can be seen as elements of the ring $\mathcal{D}(\mathcal{T}_\circ) \rtimes \mathbb{C}W$. Put $H_{\kappa, \circ} = A_\circ \otimes_A H_\kappa$. There exists a unique algebra structure on $H_{\kappa, \circ}$ extending H_κ .

Proposition 6.1. *Let $\kappa \in \mathbb{C}^\times$. There exists a unique algebra isomorphism $H_{\kappa, \circ} \rightarrow \mathcal{D}(\mathcal{T}_\circ) \rtimes \mathbb{C}W$ such that $y_i \mapsto T_i$, $w \mapsto w$, $f \mapsto f$ for all $i \in [1, n]$, $w \in W$ and $f \in A_\circ$.*

For an H_κ -module M , set $M_\circ = A_\circ \otimes_A M$. Then via Proposition 6.1, we have a structure of $\mathcal{D}(\mathcal{T}_\circ) \rtimes \mathbb{C}W$ -module on M_\circ ; namely, M_\circ admits a W -equivariant integrable (algebraic) connection

$$\nabla_i = \kappa^{-1} \left\{ y_i - \sum_{1 \leq j < i} \frac{z_j}{z_i - z_j} (1 - s_\alpha) - \sum_{i < j \leq n} \frac{z_i}{z_i - z_j} (1 - s_\alpha) - (i - 1) \right\}.$$

Now consider the case where $M = \mathcal{C}(\widehat{L}_\ell^\dagger(\mu), \widehat{L}_\ell(\lambda)) = \mathcal{L}_\kappa(\lambda, \mu)$ with $\lambda, \mu \in X_m^+(\ell)$. Then it follows that the connection given above has regular singularities along Δ , and hence $\mathcal{L}_\kappa(\lambda, \mu)_\circ$ is a projective A_\circ -module, or geometrically, a vector bundle over \mathcal{T}_\circ of finite rank ([GGOR, VV]).

For $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{T}_o$, let \mathbb{C}_ξ denote the one-dimensional right module of A_o given by the evaluation at ξ . It follows that the space $\mathbb{C}_\xi \otimes_{A_o} (\mathcal{L}_\kappa(\lambda, \mu)_o)$ is isomorphic to with “the space of conformal coinvariants”

$$\left(\widehat{L}_\ell(\mu) \otimes \widehat{L}_\ell(\nu_1)^{\otimes n} \otimes \widehat{L}_\ell(\lambda^\dagger) \right) / \mathfrak{g}_{(0, \xi, \infty)} \left(\widehat{L}_\ell(\mu) \otimes \widehat{L}_\ell(\nu_1)^{\otimes n} \otimes \widehat{L}_\ell(\lambda^\dagger) \right),$$

where $\nu_1 = (1, 0, \dots, 0) \in X_m^+(\ell)$ (the highest weight of the vector representation E), $\lambda^\dagger = -w_0(\lambda)$ with w_0 being the longest element of W , and $\mathfrak{g}_{(0, \xi, \infty)}$ denotes the Lie algebra of \mathfrak{g} -valued algebraic functions on $\mathbb{P}^1 \setminus \{0, \xi_1, \dots, \xi_n, \infty\}$, which acts on $\widehat{L}_\ell(\mu) \otimes \widehat{L}_\ell(\nu_1)^{\otimes n} \otimes \widehat{L}_\ell(\lambda^\dagger)$ through the Laurent expansion at each points. (See e.g. [BK] for a precise definition.)

Therefore it follows that the vector bundle $\mathcal{L}_\kappa(\lambda, \mu)_o$ is equivalent to the vector bundle of conformal coinvariants (the dual of the vector bundle of conformal blocks in the sense of [TUY, BK]). Moreover, the connection $\{\nabla_i\}$ on $\mathcal{L}_\kappa(\lambda, \mu)_o$ given via Proposition 6.1 coincides with the Knizhnik-Zamolodchikov connection on the vector bundle of conformal coinvariants.

7. WEIGHT DECOMPOSITION OF SYMMETRIC PART

For an H_κ -module M , put

$$(7.1) \quad M^W = \{v \in M \mid wv = v \ \forall w \in W\},$$

on which the algebra $H_\kappa^W = \{u \in H_\kappa \mid wuw^{-1} = u\}$ acts. The algebra H_κ^W is called *the zonal spherical algebra* and it contains a subalgebra $S(V)^W$, which coincides with the center of the degenerate affine Hecke algebra H^{aff} .

For $\zeta \in V^*$, let χ_ζ denote the image of the projection to the quotient space $W \setminus V^*$. Identify $W \setminus V^*$ with the set $\text{Hom}_{\text{algebra}}(S(V)^W, \mathbb{C})$ of characters, and set

$$M_{[\zeta]}^W = \{v \in M^W \mid \xi v = \chi_\zeta(\xi)v \ \forall \xi \in S(V)^W\}.$$

In the sequel, we will give a decomposition of $\mathcal{L}_\kappa(\lambda, \mu)^W$ into weight spaces with respect to $S(V)^W$.

Let $\lambda, \mu \in X_m^+$ such that $\lambda - \mu \models n$. Let λ/μ denote the skew Young diagram associated with (λ, μ) :

$$(7.2) \quad \lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], b \in [\mu_a + 1, \lambda_a]\}.$$

Let T be a tableau on the diagram λ/μ ; namely T is a bijection from λ/μ to $[1, n]$. Then it determines the sequence $\{\lambda_T^{(i)}\}_{i \in [0, n]}$ in X_m by the condition $\lambda_T^{(0)} = \mu$ and $\lambda_T^{(i)} / \lambda_T^{(i-1)} = T^{-1}(i)$ ($i \in [1, n]$).

Let $\ell \in \mathbb{Z}_{\geq 0}$. A tableau T is called an ℓ -restricted standard tableau if $\lambda_T^{(i)} \in X_m^+(\ell)$ for all $i \in [1, n]$. Let $\text{St}_{(\ell)}(\lambda, \mu)$ denote the set of ℓ -restricted tableaux on λ .

Let $T \in \text{St}_{(\ell)}(\lambda, \mu)$. For $i \in [1, n]$, define

$$(7.3) \quad h_i(T) = \begin{cases} 1 & \text{if } a < a', \\ 0 & \text{if } a \geq a', \end{cases}$$

where $T(a, b) = i$ and $T(a', b') = i + 1$. Define

$$(7.4) \quad \eta_T = \sum_{i \in [1, n]} \left(\sum_{j < i} h_j(T) \right) x_i \in P.$$

Define $\zeta_T \in V^*$ by $\zeta_T(y_i) = b - a$ when $T(a, b) = i$.

From the weight decomposition of $\mathcal{L}_\kappa(\lambda, \mu)$ (Theorem A.3) with respect to $S(V)$, we have

Theorem 7.1. (*Conjecture 6.1.1 in [AST]*) Let $\lambda, \mu \in X_m^+(\ell)$ such that $\lambda - \mu \models n$. Then

$$\mathcal{L}_\kappa(\lambda, \mu)^W = \bigoplus_{\nu \in P^-} \bigoplus_{T \in \text{St}_{(\ell)}(\lambda/\mu)} \mathcal{L}_\kappa(\lambda, \mu)_{[\zeta_T + \kappa(\nu + \eta_T)]}^W,$$

where $P^- = \{\zeta \in P \mid \langle \zeta \mid \alpha_i^\vee \rangle \leq 0 \ \forall i \in [1, n-1]\}$, and

$$\dim \mathcal{L}_\kappa(\lambda, \mu)_{[\zeta_T + \kappa(\nu + \eta_T)]}^W = 1$$

for all $\nu \in P^-$ and $T \in \text{St}_{(\ell)}(\lambda/\mu)$.

8. q -DIMENSION FORMULA

Put $\partial = \kappa^{-1} \sum_{i \in [1, n]} y_i \in S(V)^W$. Then ∂ satisfies the relation

$$[\partial, z_i] = \kappa z_i, \quad [\partial, w] = 0$$

for all $i \in [1, n]$ and $w \in W$.

Our next purpose is to give a q -dimension formula for $\mathcal{L}_\kappa(\lambda, \mu)^W$ with respect to the grading operator ∂ . To this end, we need to introduce the “polynomial part” of $\mathcal{L}_\kappa(\lambda, \mu)$ following [AST].

Define a subalgebra $H_\kappa^{\geq 0}$ of H_κ by

$$H_\kappa^{\geq 0} = \mathbb{C}P^{\geq 0} \cdot \mathbb{C}W \cdot S(V),$$

where $P^{\geq 0} = \bigoplus_{i \in [1, n]} \mathbb{Z}_{\geq 0} x_i$.

Let $\kappa \in \mathbb{Z}_{\geq 1}$ and let $\lambda, \mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$. Recall that the induced module $\mathcal{M}_\kappa(\lambda, \mu)$ is generated by the cyclic vector $1_{\lambda, \mu}$. We denote by $1_{\lambda, \mu}^-$ its image under the projection $\mathcal{M}_\kappa(\lambda, \mu) \rightarrow \mathcal{L}_\kappa(\lambda, \mu)$. Note that $1_{\lambda, \mu}^- \neq 0$. Define the *polynomial part* of $\mathcal{L}_\kappa(\lambda, \mu)$ by $\mathcal{L}_\kappa^{\geq 0}(\lambda, \mu) = H_\kappa^{\geq 0} 1_{\lambda, \mu}^-$, which is an $H_\kappa^{\geq 0}$ -submodule of $\mathcal{L}_\kappa(\lambda, \mu)$.

Put $\mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)_{(k)}^W = \{v \in \mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)^W \mid \partial v = kv\}$. Then we have $\dim \mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)_{(k)}^W < \infty$ and $\mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)^W = \bigoplus_{k \in \mathbb{Z}} \mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)_{(k)}^W$. Define

$$\dim_q \mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)^W = \sum_{d \in \mathbb{Z}} q^d \dim \mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)_{(d)}^W.$$

Set

$$(8.1) \quad h(T) = \kappa \langle \eta_T \mid \partial \rangle = \sum_{i \in [1, n]} (n - i) h_i(T).$$

From Theorem 7.1, we have

Theorem 8.1. *Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $\lambda, \mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$. Then*

$$(8.2) \quad \dim_q \mathcal{L}_\kappa^{\geq 0}(\lambda, \mu)^W = \frac{q^{\Delta_\lambda - \Delta_\mu}}{(q)_n} F_{\lambda/\mu}^{(\ell)}(q).$$

Here $\Delta_\lambda = \frac{1}{2\kappa}((\lambda, \lambda) + 2(\rho, \lambda))$, $(q)_n = (1 - q)(1 - q^2) \dots (1 - q^n)$ and $F_{\lambda/\mu}^{(\ell)}(q)$ is a polynomial of q given by

$$(8.3) \quad F_{\lambda/\mu}^{(\ell)}(q) = \sum_{T \in \text{St}_{(\ell)}(\lambda/\mu)} q^{h(T)}.$$

Remark 8.2. If ℓ is large enough then $F_{\lambda/\mu}^{(\ell)}(q)$ coincides with the Kostka polynomial $K_{(\lambda/\mu)'(1^n)}(q)$ associated to the conjugate $(\lambda/\mu)'$ of λ/μ . Hence our polynomial $F_{\lambda/\mu}^{(\ell)}(q)$ is an ℓ -restricted version of the Kostka polynomial (cf. [FJKLM]).

Remark 8.3. A bosonic formula for $F_{\lambda/\mu}^{(\ell)}(q)$ is known (Theorem 6.2.4 in [AST]), and Theorem 8.1 is equivalent to the formula in Conjecture 6.1.1 in [AST]. Note also that the bosonic formula suggests the existence of the BGG type resolution of $\mathcal{L}_\kappa(\lambda, \mu)$.

9. RATIONAL ANALOGUE

For a $\mathfrak{g}[t]$ -module N , set

$$\begin{aligned} \tilde{\mathcal{C}}(M) &= E[z_1] \otimes \dots \otimes E[z_n] \otimes N \\ \mathcal{C}(N) &= \tilde{\mathcal{C}}(N)/\mathfrak{g}[t]\tilde{\mathcal{C}}(N), \end{aligned}$$

where $E[z] = E \otimes \mathbb{C}[z]$. The analogous construction gives on $\mathcal{C}(N)$ an action of the rational Cherednik algebra H_κ^{rat} ([EG]), which can be

defined as the subalgebra of H_κ generated by the subalgebra $\mathbb{C}[\underline{z}] \cdot \mathbb{C}W$ and the following (pairwise commutative) elements

$$(9.1) \quad u_i = z_i^{-1} \left(y_i - \sum_{j < i} s_{ij} \right) \quad (i \in [1, n])$$

as pointed out in [S2].

It follows for $\lambda \in X_m^+(\ell)$ that $\mathcal{C}(\widehat{M}_\ell^\dagger(\lambda))$ is isomorphic to some induced module, and $\mathcal{C}(\widehat{L}_\ell^\dagger(\lambda))$ is isomorphic to the unique simple quotient of $\mathcal{C}(\widehat{M}_\ell^\dagger(\lambda))$, which we denote by $\mathcal{L}_\kappa(\lambda)$.

Let $\mathbf{0} = (0, \dots, 0) \in X_m^+(\ell)$. Then it follows that the polynomial part $\mathcal{L}_\kappa^{\geq 0}(\lambda, \mathbf{0})$ of the H_κ -module $\mathcal{L}_\kappa(\lambda, \mathbf{0})$ is an H_κ^{rat} -submodule and it is isomorphic to $\mathcal{L}_\kappa(\lambda)$. This leads the q -dimension formula

$$(9.2) \quad \dim_q \mathcal{L}_\kappa(\lambda)^W = \frac{q^{\Delta_\lambda}}{(q)_n} F_\lambda^{(\ell)}(q).$$

Remark 9.1. It can be seen that the *Knizhnik-Zamolodchikov functor* investigated in [GGOR] transforms the irreducible representations $\mathcal{L}_\kappa(\lambda)$ for $\lambda \in X_m^+(\ell)$ to Wenzl's representations [W] of the affine Hecke algebra (cf. [TK]).

APPENDIX A. TABLEAUX ON PERIODIC DIAGRAMS AND REPRESENTATIONS OF THE DEGENERATE DAHA

We will review the theory of tableaux representations for H_κ , which is investigated in [SV] for the double affine Hecke algebra.

Fix $\kappa \in \mathbb{Z}_{\geq 1}$. Let $m \in \mathbb{Z}_{\geq 1}$.

For $\lambda, \mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$, we introduce the following subsets of $\mathbb{Z} \times \mathbb{Z}$:

$$(A.1) \quad \lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], b \in [\mu_a + 1, \lambda_a]\},$$

$$(A.2) \quad \widehat{\lambda/\mu} = \{(a, b) + k(m, -\kappa + m) \in \mathbb{Z} \times \mathbb{Z} \mid (a, b) \in \lambda/\mu, k \in \mathbb{Z}\}.$$

The set $\widehat{\lambda/\mu}$ is called the *periodic skew diagram* of period $(m, -\kappa + m)$ associated with (λ, μ) . The following is called the skew property:

Lemma A.1. *Let $(a, b), (a', b') \in \widehat{\lambda/\mu}$. If $a' - a \in \mathbb{Z}_{\geq 0}$ and $b' - b \in \mathbb{Z}_{\geq 0}$ then $(a, b'), (a', b) \in \widehat{\lambda/\mu}$.*

A *tableau* T on $\widehat{\lambda/\mu}$ is by definition a bijection $\widehat{\lambda/\mu} \rightarrow \mathbb{Z}$ satisfying $T(a + m, b - \kappa + m) = T(a, b) + n$ for all $(a, b) \in \widehat{\lambda/\mu}$.

A tableau T is called a *standard tableau* if

$$T(a, b + 1) > T(a, b)$$

for any $(a, b), (a, b+1) \in \widehat{\lambda/\mu}$, and if

$$T(a+1, b) > T(a, b)$$

for any $(a, b), (a+1, b) \in \widehat{\lambda/\mu}$. Let $\text{Tab}(\widehat{\lambda/\mu})$ and $\text{St}(\widehat{\lambda/\mu})$ denote the set of tableaux and the set of standard tableaux on $\widehat{\lambda/\mu}$ respectively.

Define the elements $\pi = \tau_{x_1} s_1 s_2 \cdots s_{n-1}$ and $s_0 = \tau_{\alpha_{1n}} s_{1n}$ of the group $\widehat{W} = P \rtimes W$. Then $\{s_0, s_1, \dots, s_{n-1}, \pi\}$ is a generator of the group \widehat{W} .

Define the action of \widehat{W} on the set \mathbb{Z} of integers by

$$(A.3) \quad s_i(j) = \begin{cases} j+1 & \text{for } j \equiv i \pmod{n}, \\ j-1 & \text{for } j \equiv i+1 \pmod{n}, \\ j & \text{for } j \not\equiv i, i+1 \pmod{n}, \end{cases}$$

$$(A.4) \quad \tau_{x_i}(j) = \begin{cases} j+n & \text{for } j \equiv i \pmod{n}, \\ j & \text{for } j \not\equiv i \pmod{n}. \end{cases}$$

Observe that $\pi(j) = j+1$ for all j .

For $T \in \text{Tab}(\widehat{\lambda/\mu})$ and $w \in \widehat{W}$, the map $wT : \widehat{\lambda/\mu} \rightarrow \mathbb{Z}$ given by

$$(wT)(u) = w(T(u)) \quad (u \in \widehat{\lambda/\mu})$$

is also a tableau on $\widehat{\lambda/\mu}$, and the assignment $T \mapsto wT$ gives an action of \widehat{W} on $\text{Tab}(\widehat{\lambda/\mu})$, which preserves $\text{St}(\widehat{\lambda/\mu})$. It is easy to see that the assignment $w \mapsto wT$ gives a one-to-one correspondence $\widehat{W} \xrightarrow{\sim} \text{Tab}(\widehat{\lambda/\mu})$.

Define the map $C : \widehat{\lambda/\mu} \rightarrow \mathbb{Z}$ by $C(a, b) = b - a$, and define $C_T : \mathbb{Z} \rightarrow \mathbb{Z}$ by $C_T(i) = C(T^{-1}(i))$ for $T \in \text{St}(\widehat{\lambda/\mu})$. Define $\zeta_T \in V^*$ by $\langle \zeta_T | y_i \rangle = C_T(i)$ ($i \in [1, n]$).

The following lemma follows from the skew property and the definition of the standard tableaux:

Lemma A.2. *Let $T \in \text{St}(\widehat{\lambda/\mu})$ and $i \in [0, n-1]$.*

- (i) $C_T(i) - C_T(i+1) \neq 0$.
- (ii) $s_i T \in \text{St}(\widehat{\lambda/\mu})$ if and only if $C_T(i) - C_T(i+1) \notin \{-1, 1\}$.

Now, we introduce the tableaux representation associated with $\widehat{\lambda/\mu}$. Let $\mathcal{V}_\kappa(\widehat{\lambda/\mu})$ be the vector space with the basis $\{v_T\}_{T \in \text{St}(\widehat{\lambda/\mu})}$:

$$\mathcal{V}_\kappa(\widehat{\lambda/\mu}) = \bigoplus_{T \in \text{St}(\widehat{\lambda/\mu})} \mathbb{C} v_T.$$

By Lemma A.2 and induction argument, we have

Theorem A.3. (Theorem 3.16, Theorem 3.17 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let $\lambda, \mu \in X_m^+(\kappa - m)$ such that $\lambda - \mu \models n$.

(i) There exists a unique H_κ -module structure on $\mathcal{V}_\kappa(\widehat{\lambda/\mu})$ such that

$$y_i v_T = C_T(i) v_T \quad (i \in [1, n]),$$

$$\pi v_T = v_{\pi T},$$

$$s_i v_T = \begin{cases} \frac{1+a_i}{a_i} v_{s_i T} - \frac{1}{a_i} v_T & \text{if } s_i T \in \text{St}(\widehat{\lambda/\mu}) \\ -\frac{1}{a_i} v_T & \text{if } s_i T \notin \text{St}(\widehat{\lambda/\mu}) \end{cases} \quad (i \in [0, n-1]),$$

where $a_i = C_T(i) - C_T(i+1) \neq 0$ (by Lemma A.2).

(ii) $\mathcal{V}_\kappa(\lambda, \mu) = \oplus_{T \in \text{St}(\widehat{\lambda/\mu})} \mathcal{V}_\kappa(\lambda, \mu)_{\zeta_T}$, and $\mathcal{V}_\kappa(\lambda, \mu)_{\zeta_T} = \mathbb{C} v_T$ for all $T \in \text{St}(\widehat{\lambda/\mu})$.

(iii) The H_κ -module $\mathcal{V}_\kappa(\widehat{\lambda/\mu})$ is irreducible.

(iv) $\mathcal{V}_\kappa(\widehat{\lambda/\mu}) \cong \mathcal{L}_\kappa(\lambda, \mu)$.

The following result is also announced in [C2]:

Theorem A.4. (Theorem 3.19 in [SV]) Let $\kappa \in \mathbb{Z}_{\geq 1}$. Let L be an irreducible H_κ -module such that $L = \oplus_{\zeta \in P} L_\zeta$. Then there exist $m \in [1, n]$ and $\lambda, \mu \in X_m^+(\kappa - m)$ with $\lambda - \mu \models n$ such that $L \cong \mathcal{V}_\kappa(\widehat{\lambda/\mu})$.

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